



K–8 Publishers’ Criteria for the Common Core State Standards for Mathematics

These Standards are not intended to be new names for old ways of doing business. They are a call to take the next step. ... It is time to recognize that standards are not just promises to our children, but promises we intend to keep.

–CCSSM, p. 5

The Common Core State Standards were developed through a bipartisan, state-led initiative spearheaded by state superintendents and state governors. The Standards reflect the collective expertise of hundreds of teachers, education researchers, mathematicians, and state content experts from across the country. The Standards build on the best of previous state standards plus a large body of evidence from international comparisons and domestic reports and recommendations to define a sturdy staircase to college and career readiness. Most states have now adopted the Standards to replace previous expectations in English language arts/literacy and mathematics.

Standards by themselves cannot raise achievement. Standards don’t stay up late at night working on lesson plans, or stay after school making sure every student learns—it’s teachers who do that. And standards don’t implement themselves. Education leaders from the state board to the building principal must make the Standards a reality in schools. Publishers too have a crucial role to play in providing the tools that teachers and students need to meet higher standards. This document, developed by the CCSSM writing team with review and collaboration from partner organizations, individual experts, and districts using the criteria, aims to support faithful CCSSM implementation by providing criteria for materials aligned to the Common Core State Standards for Mathematics. States, districts, and publishers can use these criteria to develop, evaluate, or purchase aligned materials, or to supplement or modify existing materials to remedy weaknesses.

How should alignment be judged? Traditionally, judging alignment has been approached as a crosswalking exercise. But crosswalking can result in large percentages of “aligned content” while obscuring the fact that the materials in question align not at all to the letter or the spirit of the standards being implemented. These criteria are an attempt to sharpen the alignment question and make alignment and misalignment more clearly visible.

These criteria were developed from the perspective that publishers and purchasers are equally responsible for fixing the materials market. Publishers cannot deliver focus to buyers who only ever complain about what has been left out, yet never complain about what has crept in. More generally, publishers cannot invest in quality if the market doesn’t demand it of them nor reward them for producing it.

The K–8 Publishers’ Criteria are structured as follows:

- I. Focus, Coherence, and Rigor in the Common Core State Standards for Mathematics
- II. Criteria for Materials and Tools Aligned to the K–8 Standards
- III. Appendix: “The Structure is the Standards”

I. Focus, Coherence, and Rigor in the Common Core State Standards for Mathematics

Less topic coverage can be associated with higher scores on those topics covered because students have more time to master the content that is taught.

—Ginsburg et al., 2005, *Reassessing U.S. International Mathematics Performance: New Findings from the 2003 TIMSS and PISA*

This finding that postsecondary instructors target fewer skills as being of high importance is consistent with recent policy statements and findings raising concerns that some states require too many standards to be taught and measured, rather than focusing on the most important state standards for students to attain. ...

Because the postsecondary survey results indicate that a more rigorous treatment of fundamental content knowledge and skills needed for credit-bearing college courses would better prepare students for postsecondary school and work, states would likely benefit from examining their state standards and, where necessary, reducing them to focus only on the knowledge and skills that research shows are essential to college and career readiness and postsecondary success. ...

—ACT National Curriculum Survey 2009

Because the mathematics concepts in [U.S.] textbooks are often weak, the presentation becomes more mechanical than is ideal. We looked at both traditional and non-traditional textbooks used in the US and found conceptual weakness in both.

—Ginsburg et al., 2005, cited in CCSSM, p. 3

...[B]ecause conventional textbook coverage is so fractured, unfocused, superficial, and unprioritized, there is no guarantee that most students will come out knowing the essential concepts of algebra.

—Wiggins, 2012¹

For years national reports have called for greater focus in U.S. mathematics education. TIMSS and other international studies have concluded that mathematics education in the United States is a mile wide and an inch deep. A mile-wide inch-deep curriculum translates to less time per topic. Less time means less depth and moving on without many students. In high-performing countries, strong foundations are laid and then further knowledge is built on them; the design principle in those countries is focus with coherent progressions. The U.S. has lacked such discipline and patience.

There is evidence that state standards have become somewhat more focused over the past decade. But in the absence of standards shared across states, instructional materials have not followed suit. Moreover, prior to the Common Core, state standards were making little progress in terms of coherence: states were not fueling achievement by organizing math so that the subject makes sense.

With the advent of the Common Core, a decade's worth of recommendations for greater focus and coherence finally have a chance to bear fruit. Focus and coherence are the two major evidence-based design principles of the Common Core State Standards for Mathematics.² These principles are meant to fuel greater achievement in a deep and rigorous curriculum, one in which students acquire

¹ From <http://grantwiggins.wordpress.com/2012/02/01/a-postscript-to-my-comment-about-kids-having-trouble-with-the-distributive-property>.

² For some of the sources of evidence consulted during the standards development process, see pp. 91–93 of CCSSM.

conceptual understanding, procedural skill and fluency, and the ability to apply mathematics to solve problems. Thus, the implications of the standards for mathematics education could be summarized briefly as follows:

Focus: focus strongly where the standards focus

Coherence: think across grades, and link to major topics in each grade

Rigor: in major topics, pursue with equal intensity

- conceptual understanding,
- procedural skill and fluency, and
- applications

Focus

Focus means significantly narrowing the scope of content in each grade so that students achieve at higher levels and experience more deeply that which remains.

We have come to see “narrowing” as a bad word—and it is a bad word, if it means cutting arts programs and language programs. But math has swelled in this country. The standards are telling us that math actually needs to lose a few pounds.

The strong focus of the Standards in early grades is arithmetic along with the components of measurement that support it. That includes the concepts underlying arithmetic, the skills of arithmetic computation, and the ability to apply arithmetic to solve problems and put arithmetic to engaging uses. Arithmetic in the K–5 standards is an important life skill, as well as a thinking subject and a rehearsal for algebra in the middle grades.

Focus remains important through the middle and high school grades in order to prepare students for college and careers. National surveys have repeatedly concluded that postsecondary instructors value greater mastery of a smaller set of prerequisites over shallow exposure to a wide array of topics, so that students can build on what they know and apply what they know to solve substantial problems.

During the writing of the Standards, the writing team often received feedback along these lines: “I love the focus of these standards! Now, if we could just add one or two more things...” But focus compromised is no longer focus at all. Faithfully implementing the standards requires moving some topics traditionally taught in earlier grades up to higher grades entirely, sometimes to much higher grades. “Teaching less, learning more” can seem like hard medicine for an educational system addicted to coverage. But remember that the goal of focus is to make good on the ambitious promise the states have made to their students by adopting the Standards: greater achievement at the college- and career-ready level, greater depth of understanding of mathematics, and a rich classroom environment in which reasoning, sense-making, applications, and a range of mathematical practices all thrive. None of this is realistic in a mile-wide, inch-deep world.

Both of the assessment consortia have made the focus, coherence, and rigor of the Standards central to their assessment designs.³ Choosing materials that also embody the Standards will be essential for giving teachers and students the tools they need to build a strong mathematical foundation and succeed on the coming aligned exams.

Coherence

Coherence is about making math make sense. Mathematics is not a list of disconnected tricks or mnemonics. It is an elegant subject in which powerful knowledge results from reasoning with a small number of principles such as place value and properties of operations.⁴ The Standards define progressions of learning that leverage these principles as they build knowledge over the grades.⁵

Coherence has to do with connections between topics. Vertical connections are crucial: these are the links from one grade to the next that allow students to progress in their mathematical education. For example, a kindergarten student might add two numbers using a “count all” strategy, but grade 1 students are expected to use “counting on” and more sophisticated strategies. It is critical to think across grades and examine the progressions in the standards to see how major content develops over time.

The Standards do not specify the progression of material within a single grade, but coherence across grades also depends on having careful, deliberate, and progressive development of ideas within each grade. Some examples of this can be seen in the *Progressions* documents.⁶ For example, it would not make sense to address cluster 8.EE.B (understanding the connections between proportional relationships, lines, and linear equations) before addressing triangle similarity, as ideas of triangle similarity underlie the very definition of the slope of a line in the coordinate plane.

Connections at a single grade level can be used to improve focus, by closely linking secondary topics to the major work of the grade. For example, in grade 3, bar graphs are not “just another topic to cover.” Rather, the standard about bar graphs asks students to use information presented in bar graphs to solve word problems using the four operations of arithmetic. Instead of allowing bar graphs to detract from the focus on arithmetic, the Standards are showing how bar graphs can be positioned in support of the major work of the grade. In this way coherence can support focus.

Materials cannot match the contours of the Standards by approaching each individual content standard as a separate event. Nor can materials align to the Standards by approaching each individual grade as a separate event. From the Appendix: “The standards were not so much assembled out of topics as woven out of progressions. Maintaining these progressions in the implementation of the standards will be important for helping all students learn mathematics at a higher level. ... For example, the properties of operations, learned first for simple whole numbers, then in later grades extended to fractions, play a central role in understanding operations with negative numbers,

³ See the Smarter/Balanced content specification and item development specifications, and the PARCC Model Content Framework and item development ITN. Complete information about the consortia can be found at www.smarterbalanced.org and www.parcconline.org.

⁴ For some remarks by Phil Daro on this theme, see the excerpt at <http://vimeo.com/achievethecore/darofocus>, and/or the full video available at <http://commoncoretools.me/2012/05/21/phil-daro-on-learning-mathematics-through-problem-solving/>.

⁵ For more information on progressions in the Standards, see <http://ime.math.arizona.edu/progressions>.

⁶ <http://ime.math.arizona.edu/progressions>

expressions with letters and later still the study of polynomials. As the application of the properties is extended over the grades, an understanding of how the properties of operations work together should deepen and develop into one of the most fundamental insights into algebra. The natural distribution of prior knowledge in classrooms should not prompt abandoning instruction in grade level content, but should prompt explicit attention to connecting grade level content to content from prior learning. To do this, instruction should reflect the progressions on which the CCSSM are built.”

“Fragmenting the Standards into individual standards, or individual bits of standards, ... produces a sum of parts that is decidedly less than the whole” (Appendix). Breaking down standards poses a threat to the focus and coherence of the Standards. It is sometimes helpful or necessary to isolate a part of a compound standard for instruction or assessment, but not always, and not at the expense of the Standards as a whole. A drive to break the Standards down into ‘microstandards’ risks making the checklist mentality even worse than it is today. Microstandards would also make it easier for microtasks and microlessons to drive out extended tasks and deep learning. Finally, microstandards could allow for micromanagement: Picture teachers and students being held accountable for ever more discrete performances. If it is bad today when principals force teachers to write the standard of the day on the board, think of how it would be if every single standard turns into three, six, or a dozen or more microstandards. If the Standards are like a tree, then microstandards are like twigs. You can’t build a tree out of twigs, but you can use twigs as kindling to burn down a tree.

Rigor

To help students meet the expectations of the Standards, educators will need to pursue, with equal intensity, three aspects of rigor in the major work of each grade: (1) conceptual understanding, (2) procedural skill and fluency, and (3) applications. The word “rigor” isn’t a code word for just one of these three; rather, it means equal intensity in all three. The word “understand” is used in the Standards to set explicit expectations for conceptual understanding, the word “fluently” is used to set explicit expectations for fluency, and the phrase “real-world problems” and the star symbol (★) are used to set expectations and flag opportunities for applications and modeling. (Modeling is a Standard for Mathematical Practice as well as a content category in High School.)

To date, curricula have not always been balanced in their approach to these three aspects of rigor. Some curricula stress fluency in computation without acknowledging the role of conceptual understanding in attaining fluency and making algorithms more learnable. Some stress conceptual understanding without acknowledging that fluency requires separate classroom work of a different nature. Some stress pure mathematics without acknowledging that applications can be highly motivating for students and that a mathematical education should make students fit for more than just their next mathematics course. At another extreme, some curricula focus on applications without acknowledging that math doesn’t teach itself.

The Standards do not take sides in these ways, but rather they set high expectations for all three components of rigor in the major work of each grade. Of course, that makes it necessary that we focus—otherwise we are asking teachers and students to do more with less.

II. Criteria for Materials and Tools Aligned to the Standards

The single most important flaw in United States mathematics instruction is that the curriculum is “a mile wide and an inch deep.” This finding comes from research comparing the U.S. curriculum to high performing countries, surveys of college faculty and teachers, the National Math Panel, the Early Childhood Learning Report, and all the testimony the CCSS writers heard. The standards are meant to be a blueprint for math instruction that is more focused and coherent. ... Crosswalks and alignments and pacing plans and such cannot be allowed to throw away the focus and coherence and regress to the mile-wide curriculum.

—Daro, McCallum, and Zimba, 2012 (from the Appendix)

Using the criteria

One approach to developing a document such as this one would have been to develop a separate criterion for each mathematical topic approached in deeper ways in the Standards, a separate criterion for each of the Standards for Mathematical Practice, etc. It is indeed necessary for textbooks to align to the Standards in detailed ways. However, enumerating those details here would have led to a very large number of criteria. Instead, the criteria use the Standards’ focus, coherence, and rigor as the main themes. In addition, this document includes a section on indicators of quality in materials and tools, as well as a criterion for the mathematics and statistics in instructional resources for science and technical subjects. Note that the criteria apply to materials and tools, not to teachers or teaching.

The criteria can be used in several ways:

- *Informing purchases and adoptions.* Schools or districts evaluating materials and tools for purchase can use the criteria to test claims of alignment. States reviewing materials and tools for adoption can incorporate these criteria into their rubrics. Publishers currently modifying their programs, or designing new materials and tools, can use the criteria to shape these projects.
- *Working with previously purchased materials.* Most existing materials and tools likely fail to meet one or more of these criteria, even in cases where alignment to the Standards is claimed. But the pattern of failure is likely to be informative. States and districts need not wait for “the perfect book” to arrive, but can use the criteria now to carry out a thoughtful plan to modify or combine existing resources in such a way that students’ actual learning experiences approach the focus, coherence, and rigor of the Standards. Publishers can develop innovative materials and tools specifically aimed at addressing identified weaknesses of widespread textbooks or programs.
- *Guiding the development of materials.* Publishers currently modifying their programs and designers of new materials and tools can use the criteria to shape these projects.
- *Professional development.* The criteria can be used to support activities that help communicate the shifts in the Standards. For example, teachers can analyze existing materials to reveal how they treat the major work of the grade, or assess how well materials attend to the three aspects of rigor, or determine which problems are key to developing the ideas and skills of the grade.

In all these cases, it is recommended that the criteria for focus be attended to first. By attending first to focus, coherence and rigor may realistically develop.

The Standards do not dictate the acceptable forms of instructional resources—to the contrary, they are a historic opportunity to raise student achievement through innovation. Materials and tools of very different forms can meet the criteria, including workbooks, multi-year programs, and targeted interventions. For example, materials and tools that treat a single important topic or domain might be valuable to consider.

Alignment for digital and online materials and tools. Digital materials offer substantial promise for conveying mathematics in new and vivid ways and customizing learning. In a digital or online format, diving deeper and reaching back and forth across the grades is easy and often useful. That can enhance focus and coherence. But if such capabilities are poorly designed, focus and coherence could also be diminished. In a setting of dynamic content navigation, the navigation experience must preserve the coherence of Standards clusters and progressions while allowing flexibility and user control: Users can readily see where they are with respect to the structure of the curriculum and its basis in the Standards' domains, clusters and standards.

Digital materials that are smaller than a course can be useful. The smallest granularity for which they can be properly evaluated is a cluster of standards. These criteria can be adapted for clusters of standards or progressions within a cluster, but might not make sense for isolated standards.

Special populations. As noted in the Standards (p. 4),

All students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills necessary in their post-school lives. The Standards should be read as allowing for the widest possible range of students to participate fully from the outset, along with appropriate accommodations to ensure maximum participation of students with special education needs.

Thus, **an over-arching criterion** for materials and tools is that they provide supports for special populations such as students with disabilities, English language learners,⁷ and gifted students. Designers of materials should consult accepted guidelines for providing these supports.

*

For the sake of brevity, the criteria sometimes refer to parts of the Standards using abbreviations such as 3.MD.7 (an individual content standard), MP.8 (a practice standard), 8.EE.B (a cluster heading), or 4.NBT (a domain heading). Readers of the document should have a copy of the Standards available in order to refer to the indicated text in each case.

⁷ Slides from a brief and informal presentation by Phil Daro about mathematical language and English language learners can be found at <http://db.tt/VARV3ebl>.

Criteria for Materials and Tools Aligned to the Standards

1. **Focus on Major Work: In any single grade, students and teachers using the materials as designed spend the large majority of their time on the major work of each grade.**⁸ In order to preserve the focus and coherence of the Standards, both assessment consortia have designated clusters at each grade level as major, additional, or supporting,⁹ with clusters designated as major comprising the major work of each grade. Major work is not the only work in the Standards, but materials are highly unlikely to be aligned to the Standards' focus unless they dedicate the large majority of their time¹⁰ on the major work of each grade.

This criterion also applies to digital or online materials without fixed pacing plans. Such tools are explicitly designed for focus, so that students spend the large majority of their time on the major work of each grade.

Note that an important **subset** of the major work in grades K–8 is the progression that leads toward middle-school algebra (see Table 1, next page). Materials give especially careful treatment to these clusters and their interconnections.¹¹

⁸ The materials should devote at least 65% and up to approximately 85% of the class time to the major work of the grade with Grades K–2 nearer the upper end of that range, i.e., 85%.

⁹ For cluster-level emphases at grades K–2, see <http://www.achievethecore.org/downloads/Math%20Shifts%20and%20Major%20Work%20of%20Grade.pdf>.

¹⁰ The materials should devote at least 65% and up to approximately 85% of the class time to the major work of the grade with Grades K–2 nearer the upper end of that range, i.e., 85%.

¹¹ For domain-by-domain progressions in the Standards, see <http://ime.math.arizona.edu/progressions>.

Table 1. Progress to Algebra in Grades K–8

K	1	2	3	4	5	6	7	8
Know number names and the count sequence	Represent and solve problems involving addition and subtraction		Represent & solve problems involving multiplication and division	Use the four operations with whole numbers to solve problems	Understand the place value system	Apply and extend previous understandings of multiplication and division to divide fractions by fractions		
Count to tell the number of objects	Understand and apply properties of operations and the relationship between addition and subtraction	Represent and solve problems involving addition and subtraction	Understand properties of multiplication and the relationship between multiplication and division	Generalize place value understanding for multi-digit whole numbers	Perform operations with multi-digit whole numbers and decimals to hundredths	Apply and extend previous understandings of numbers to the system of rational numbers	Apply and extend previous understanding of operations with fractions to add, subtract, multiply, and divide rational numbers	Work with radical and integer exponents
Compare numbers	Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from	Add and subtract within 20	Multiply & divide within 100	Use place value understanding and properties of operations to perform multi-digit arithmetic	Use equivalent fractions as a strategy to add and subtract fractions	Understand ratio concepts and use ratio reasoning to solve problems	Analyze proportional relationship and use them to solve real-world and mathematical problems	Understand the connections between proportional relationships, lines, and linear equations**
Work with numbers 11-19 to gain foundations for place value	Extend the counting sequence	Understand place value	Solve problems involving the four operations, and identify & explain patterns in arithmetic	Extend understanding of fraction equivalence and ordering	Apply and extend previous understandings of multiplication and division to multiply and divide fractions	Apply and extend previous understandings of arithmetic to algebraic expressions	Use properties of operations to generate equivalent expressions	Analyze and solve linear equations and pairs of simultaneous linear equations
	Understand place value	Use place value understanding and properties of operations to add and subtract	Develop understanding of fractions as numbers	Build fractions from unit fractions by applying and extending previous understandings of operations	Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition	Reason about and solve one-variable equations and inequalities	Solve real-life and mathematical problems using numerical and algebraic expressions and equations	Define, evaluate, and compare functions
	Use place value understanding and properties of operations to add and subtract	Relate addition and subtraction to length	Solve problems involving measurement and estimation of intervals of time, liquid volumes, & masses of objects	Understand decimal notation for fractions, and compare decimal fractions	Graph points in the coordinate plane to solve real-world and mathematical problems*	Represent and analyze quantitative relationships between dependent and independent variables		Use functions to model relationships between quantities
	Measure lengths indirectly and by iterating length units		Geometric measurement: understand concepts of area and relate area to multiplication and to addition					

*Indicates a cluster that is well thought of as part of a student’s progress to algebra, but that is currently not designated as Major by one or both of the assessment consortia in their draft materials. Apart from the asterisked exception, the clusters listed here are a subset of those designated as Major in both of the assessment consortia’s draft documents. ** Depends on similarity ideas from geometry to show that slope can be defined and then used to show that a linear equation has a graph which is a straight line and conversely.

2. **Focus in Early Grades: Materials do not assess any of the following topics before the grade level indicated.**

Table 2

Topic	Grade Introduced in the Standards
Probability , including chance, likely outcomes, probability models.	7
Statistical distributions , including center, variation, clumping, outliers, mean, median, mode, range, quartiles, and statistical association or trends , including two-way tables, bivariate measurement data, scatter plots, trend line, line of best fit, correlation.	6
Similarity, congruence, or geometric transformations.	8
Symmetry of shapes, including line/reflection symmetry, rotational symmetry.	4

As the second column indicates, the Standards as a whole do include the topics in Table 2—they are not being left out. However, in the coherent progression of the Standards, these topics first appear at later grades in order to establish focus. Thus, in aligned materials there are no chapter tests, unit tests, or other such assessment components that make students or teachers responsible for any of the above topics before the grade in which they are introduced in the Standards. (One way to meet this criterion is for materials to omit these topics entirely prior to the indicated grades.)

3. **Focus and Coherence through Supporting Work: Supporting content enhances focus and coherence simultaneously by engaging students in the major work of the grade.** For example, materials for K–5 generally treat data displays as an occasion for solving grade-level word problems using the four operations (see 3.MD.3);¹² materials for grade 7 take advantage of opportunities to use probability to support ratios, proportions, and percents. (This criterion does not apply in the case of targeted supplemental materials or other tools that do not include supporting content.)
4. **Rigor and Balance: Materials and tools reflect the balances in the Standards and help students meet the Standards’ rigorous expectations, by (all of the following, in the case of comprehensive materials; at least one of the following for supplemental or targeted resources):**
- Developing students’ conceptual understanding of key mathematical concepts, especially where called for in specific content standards or cluster headings.** Materials amply feature high-quality conceptual problems and questions. This includes brief conceptual problems with low computational difficulty (e.g., ‘Find a number greater than $\frac{1}{5}$ and less than $\frac{1}{4}$ ’); brief

¹² For more information about this example, see Table 1 in the *Progression* for K-3 Categorical Data and 2-5 Measurement Data, http://commoncoretools.files.wordpress.com/2011/06/ccss_progression_md_k5_2011_06_20.pdf. More generally, the *PARCC Model Content Frameworks* give examples in each grade of how to improve focus and coherence by linking supporting topics to the major work.

conceptual questions (e.g., ‘If the divisor does not change and the dividend increases, what happens to the quotient?’); and problems that involve identifying correspondences across different mathematical representations of quantitative relationships.¹³ Classroom discussion about such problems can offer opportunities to engage in mathematical practices such as constructing and critiquing arguments (MP.3). In the materials, conceptual understanding is attended to most thoroughly in those places in the content standards where explicit expectations are set for understanding or interpreting. Such problems and activities center on fine-grained mathematical concepts—place value, the whole-number product $a \times b$, the fraction a/b , the fraction product $(a/b) \times q$, expressions as records of calculations, solving equations as a process of answering a question, etc. Conceptual understanding of key mathematical concepts is thus distinct from applications or fluency work, and these three aspects of rigor must be balanced as indicated in the Standards.

- b. **Giving attention throughout the year to individual standards that set an expectation of procedural skill and fluency.** The Standards are explicit where fluency is expected. Materials in grades K–6 help students make steady progress throughout the year toward fluent (accurate and reasonably fast) computation, including knowing single-digit products and sums from memory (see, e.g., 2.OA.2 and 3.OA.7). Progress toward these goals is interwoven with students’ developing conceptual understanding of the operations in question.¹⁴ Manipulatives and concrete representations such as diagrams that enhance conceptual understanding are connected to the written and symbolic methods to which they refer (see, e.g., 1.NBT). As well, purely procedural problems and exercises are present. These include cases in which opportunistic strategies are valuable—e.g., the sum $698 + 240$ or the system $x + y = 1$, $2x + 2y = 3$ —as well as an ample number of generic cases so that students can learn and practice efficient algorithms (e.g., the sum $8767 + 2286$). Methods and algorithms are general and based on principles of mathematics, not mnemonics or tricks.¹⁵ Materials attend most thoroughly to those places in the content standards where explicit expectations are set for fluency. In higher grades, algebra is the language of much of mathematics. Like learning any language, we learn by using it. Sufficient practice with algebraic operations is provided so as to make realistic the attainment of the Standards as a whole; for example, fluency in algebra can help students get past the need to manage computational details so that they can observe structure (MP.7) and express regularity in repeated reasoning (MP.8).
- c. **Allowing teachers and students using the materials as designed to spend sufficient time working with engaging applications, without losing focus on the major work of each grade.** Materials in grades K–8 include an ample number of single-step and multi-step contextual problems that develop the mathematics of the grade, afford opportunities for practice, and

¹³ Note that for ELL students, multiple representations also serve as multiple access paths.

¹⁴ For more about how students develop fluency in tandem with understanding, see the *Progressions* for Operations and Algebraic Thinking, http://commoncoretools.files.wordpress.com/2011/05/ccss_progression_cc_0a_k5_2011_05_302.pdf and for Number and Operations in Base Ten, http://commoncoretools.files.wordpress.com/2011/04/ccss_progression_nbt_2011_04_073.pdf.

¹⁵ Non-mathematical approaches (such as the “butterfly method” of adding fractions) compromise focus and coherence and displace mathematics in the curriculum (cf. 5.NF.1). For additional background on this point, see the remarks by Phil Daro excerpted at <http://vimeo.com/achievethecore/darofocus> and/or the full video, available at <http://commoncoretools.me/2012/05/21/phil-daro-on-learning-mathematics-through-problem-solving/>.

engage students in problem solving. Materials for grades 6–8 also include problems in which students must make their own assumptions or simplifications in order to model a situation mathematically. Applications take the form of problems to be worked on individually as well as classroom activities centered on application scenarios. Materials attend thoroughly to those places in the content standards where expectations for multi-step and real-world problems are explicit. Students learn to use the content knowledge and skills specified in the content standards in applications, with particular stress on applying major work, and a preference for the more fundamental techniques from additional and supporting work. Modeling builds slowly across K–8, and applications are relatively simple in earlier grades. Problems and activities are grade-level appropriate, with a sensible tradeoff between the sophistication of the problem and the difficulty or newness of the content knowledge the student is expected to bring to bear.

Additional aspects of the Rigor and Balance Criterion:

(1) *The three aspects of rigor are not always separate in materials.* (Conceptual understanding and fluency go hand in hand; fluency can be practiced in the context of applications; and brief applications can build conceptual understanding.)

(2) *Nor are the three aspects of rigor always together in materials.* (Fluency requires dedicated practice to that end. Rich applications cannot always be shoehorned into the mathematical topic of the day. And conceptual understanding will not always come along for free unless explicitly taught.)

(3) Digital and online materials with no fixed lesson flow or pacing plan are not designed for superficial browsing but rather should be designed to instantiate the Rigor and Balance criterion.

5. Consistent Progressions: Materials are consistent with the progressions in the Standards, by (all of the following):

a. Basing content progressions on the grade-by-grade progressions in the Standards.

Progressions in materials match well with those in the Standards. Any discrepancies in content progressions enhance the required learning in each grade and are clearly aimed at helping students meet the Standards as written, rather than setting up competing requirements or effectively rewriting the standards. Comprehensive materials do not introduce gaps in learning by omitting any content that is specified in the Standards.

The basic model for grade-to-grade progression involves students making tangible progress during each given grade, as opposed to substantially reviewing then marginally extending from previous grades. Remediation may be necessary, particularly during transition years, and resources for remediation may be provided, but previous-grades review is clearly identified as such to the teacher, and teachers and students can see what their specific responsibility is for the current year.

Digital and online materials that allow students and/or teachers to navigate content across grade levels promote the Standards' coherence by tracking the structure and progressions in the Standards. For example, such materials might link problems and concepts so that teachers and students can browse a progression.

- b. **Giving all students extensive work with grade-level problems.** Differentiation is sometimes necessary, but materials often manage unfinished learning from earlier grades inside grade level work, rather than setting aside grade-level work to reteach earlier content. Unfinished learning from earlier grades is normal and prevalent; it should not be ignored nor used as an excuse for cancelling grade level work and retreating to below-grade work. (For example, the development of fluency with division using the standard algorithm in grade 6 is the occasion to surface and deal with unfinished learning about place value; this is more productive than setting aside division and backing up.) Likewise, students who are “ready for more” can be provided with problems that take grade-level work in deeper directions, not just exposed to later grades’ topics.
- c. **Relating grade level concepts explicitly to prior knowledge from earlier grades.** The materials are designed so that prior knowledge becomes reorganized and extended to accommodate the new knowledge. Grade-level problems in the materials often involve application of knowledge learned in earlier grades. Although students may well have learned this earlier content, they have not learned how it extends to new mathematical situations and applications. They learn basic ideas of place value, for example, and then extend them across the decimal point to tenths and beyond. They learn properties of operations with whole numbers, and then extend them to fractions, variables, and expressions. The materials make these extensions of prior knowledge explicit. Thus, materials routinely integrate new knowledge with knowledge from earlier grades. Note that cluster headings in the Standards sometimes signal key moments where reorganizing and extending previous knowledge is important in order to accommodate new knowledge (e.g., see the cluster headings that use the phrase “Apply and extend previous understanding”).

6. **Coherent Connections: Materials foster coherence through connections at a single grade, where appropriate and where required by the Standards, by (all of the following):**

- a. **Including learning objectives that are visibly shaped by CCSSM cluster headings.** Cluster headings function like topic sentences in a paragraph in that they state the point of, and lend additional meaning to, the individual content standards that follow. While some clusters are simply the sum of their individual standards (e.g., 8.EE.C), many are not (e.g., 8.EE.B). In the latter case, the cluster heading signals the importance of using similarity ideas from geometry to show that slope can be defined and then used to show that a linear equation has a graph which is a straight line, and conversely.

Cluster headings can also signal multi-grade progressions, by using phrases such as “Apply and extend previous understandings of [X] to do [Y].” Hence an important criterion for coherence is that some or many of the learning objectives in the materials are visibly shaped by CCSSM cluster headings. Materials do not simply treat the Standards as a sum of individual content standards and individual practice standards.

- b. **Including problems and activities that serve to connect two or more clusters in a domain, or two or more domains in a grade, in cases where these connections are natural and important.** If instruction only operates at the individual standard level, or even at the individual cluster level, then some important connections will be missed. For example, robust work in 4.NBT should sometimes or often synthesize across the clusters listed in that domain;

robust work in grade 4 should sometimes or often involve students applying their developing computation NBT skills in the context of solving word problems detailed in OA. Materials do not invent connections not explicit in the standards without first attending thoroughly to the connections that are required explicitly in the Standards (e.g., 3.MD.7 connects area to multiplication, to addition, and to properties of operations) Not everything in the standards is naturally well connected or needs to be connected (e.g., Order of Operations has essentially nothing to do with the properties of operations, and connecting these two things in a lesson or unit title is actively misleading). Instead, connections in materials are mathematically natural and important (e.g., base-ten computation in the context of word problems with the four operations), reflecting plausible direct implications of what is written in the Standards without creating additional requirements.

- c. **Preserving the focus, coherence, and rigor of the Standards even when targeting specific objectives.** Sometimes a content standard is a compound statement, such as ‘Do X and do Y.’ More intricate compound forms also exist. (For example, see A-APR.1.) It is sometimes helpful or necessary to isolate a part of a compound standard, but not always, and not at the expense of the Standards as a whole. Digital or print materials or tools are not aligned if they break down the Standards in such a way as to detract from focus, coherence, or rigor. This criterion applies to student-facing and teacher-facing materials, as well as to architectural documents or digital platforms that are meant to guide the development of student-facing or teacher-facing materials.

7. **Practice-Content Connections: Materials meaningfully connect content standards and practice standards.** “Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.” (CCSSM, p. 8.) Over the course of any given year of instruction, each mathematical practice standard is meaningfully present in the form of activities or problems that stimulate students to develop the habits of mind described in the practice standards. These practices are well-grounded in the content standards.

The practice standards are not just processes with ephemeral products (such as conversations). They also specify a set of products students are supposed to learn how to produce. Thus, students are asked to produce answers and solutions but also, in a grade-appropriate way, arguments, explanations, diagrams, mathematical models, etc.

Materials are accompanied by an analysis, aimed at evaluators, of how the authors have approached each practice standard in relation to content within each applicable grade or grade band, and provide suggestions for delivering content in ways that help students meet the practice standards in grade-appropriate ways. Materials do not treat the practice standards as static across grades or grade bands, but instead tailor the connections to the content of the grade and to grade-level-appropriate student thinking. Materials also include teacher-directed materials that explain the role of the practice standards in the classroom and in students’ mathematical development.

8. **Focus and Coherence via Practice Standards: Materials promote focus and coherence by connecting practice standards with content that is emphasized in the Standards.** Content and practice standards are not connected mechanistically or randomly, but instead support focus and

coherence. Examples: Materials connect looking for and making use of structure (MP.7) with structural themes emphasized in the standards such as properties of operations, place value decompositions of numbers, numerators and denominators of fractions, numerical and algebraic expressions, etc.; materials use repeated reasoning (MP.8) as a *tool* with which to explore content that is emphasized in the Standards. (In K-5, materials might use regularity in repetitive reasoning to shed light on, e.g., the 10×10 addition table, the 10×10 multiplication table, the properties of operations, the relationship between addition and subtraction or multiplication and division, and the place value system; in 6-8, materials might use regularity in repetitive reasoning to shed light on proportional relationships and linear functions; in high school, materials might use regularity in repetitive reasoning to shed light on formal algebra as well as functions, particularly recursive definitions of functions.)

9. **Careful Attention to Each Practice Standard: Materials attend to the full meaning of each practice standard.** For example, MP.1 does not say, “Solve problems.” Or “Make sense of problems.” Or “Make sense of problems and solve them.” It says “Make sense of problems and persevere in solving them.” Thus, students using the materials as designed build their perseverance in grade-level-appropriate ways by occasionally solving problems that require them to persevere to a solution beyond the point when they would like to give up.¹⁶ MP.5 does not say, “Use tools.” Or “Use appropriate tools.” It says “Use appropriate tools strategically.” Thus, materials include problems that reward students’ strategic decisions about how to use tools, or about whether to use them at all. MP.8 does not say, “Extend patterns.” Or “Engage in repetitive reasoning.” It says “Look for and express regularity in repeated reasoning.” Thus, it is not enough for students to extend patterns or perform repeated calculations. Those repeated calculations must lead to an insight (e.g., “When I add a multiple of 3 to another multiple of 3, then I get a multiple of 3.”). The analysis for evaluators explains how the full meaning of each practice standard has been attended to in the materials.
10. **Emphasis on Mathematical Reasoning: Materials support the Standards’ emphasis on mathematical reasoning, by (all of the following):**
- Prompting students to construct viable arguments and critique the arguments of others concerning key grade-level mathematics that is detailed in the content standards (cf. MP.3).** Materials provide sufficient opportunities for students to reason mathematically and express reasoning through classroom discussion, written work and independent thinking. Reasoning is not confined to optional or avoidable sections of the materials but is inevitable when using the materials as designed. Materials do not approach reasoning as a generalized imperative, but instead create opportunities for students to reason *about* key mathematics detailed in the content standards for the grade. Materials thus attend first and most thoroughly to those places in the content standards setting explicit expectations for

¹⁶ Curriculum designers might consider how research on motivation and character development has value for designing tools that develop students’ perseverance and other mathematical practices. For more information, see, e.g., Dweck (2008), “Mindsets and Math” (<http://opportunityequation.org/teaching-and-leadership/mindsets-math-science-achievement>); Duckworth et al. (2007), “Grit: Perseverance and Passion for Long-Term Goals” (http://www.sas.upenn.edu/~duckwort/images/publications/DuckworthPetersonMatthewsKelly_2007_PerseveranceandPassion.pdf); and <http://www.psychologicalscience.org/index.php/publications/observer/2013/april-13/true-grit.html>.

explaining, justifying, showing, or proving. Students are asked to critique given arguments, e.g., by explaining under what conditions, if any, a mathematical statement is valid. Materials develop students' capacity for mathematical reasoning in a grade-level appropriate way, with a reasonable progression of sophistication from early grades up through high school.¹⁷ Teachers and students using the materials as designed spend significant classroom time communicating reasoning (by constructing viable arguments and critiquing the arguments of others concerning key grade-level mathematics)—recognizing that learning mathematics also involves time spent working on applications and practicing procedures. Materials provide examples of student explanations and arguments (e.g., fictitious student characters might be portrayed).

- b. **Engaging students in problem solving as a form of argument.** Materials attend thoroughly to those places in the content standards that explicitly set expectations for multi-step problems; multi-step problems are not scarce in the materials. Some or many of these problems require students to devise a strategy autonomously. Sometimes the goal is the final answer alone (cf. MP.1); sometimes the goal is to lay out the solution as a sequence of well justified steps. In the latter case, the solution to a problem takes the form of a cogent argument that can be verified and critiqued, instead of a jumble of disconnected steps with a scribbled answer indicated by drawing a circle around it (cf. MP.6). Problems and activities of this nature are grade-level appropriate, with a reasonable progression of sophistication from early grades up through high school.
- c. **Explicitly attending to the specialized language of mathematics.** Mathematical reasoning involves specialized language. Therefore, materials and tools address the development of mathematical and academic language associated with the standards. The language of argument, problem solving and mathematical explanations are taught rather than assumed. Correspondences between language and multiple mathematical representations including diagrams, tables, graphs, and symbolic expressions are identified in material designed for language development. Note that variety in formats and types of representations—graphs, drawings, images, and tables in addition to text—can relieve some of the language demands that English language learners face when they have to show understanding in math.

The text is considerate of English language learners, helping them to access challenging mathematics and helping them to develop grade level language. For example, materials might include annotations to help with comprehension of words, sentences and paragraphs, and give examples of the use of words in other situations. Modifications to language do not sacrifice the mathematics, nor do they put off necessary language development.

¹⁷ As students progress through the grades, their production and comprehension of mathematical arguments evolves from informal and concrete toward more formal and abstract. In early grades students employ imprecise expressions which with practice over time become more precise and viable arguments in later grades. Indeed, the use of imprecise language is part of the process in learning how to make more precise arguments in mathematics. Ultimately, conversation about arguments helps students transform assumptions into explicit and precise claims.

A criterion for the mathematics and statistics in materials for science and technical subjects

Lack of alignment in these subjects could have the effect of compromising the focus and coherence of the mathematics Standards. Instead of reinforcing concepts and skills already carefully introduced in math class, teachers of science and technical subjects would have to teach this material in stopgap fashion. That wouldn't serve students well in any grade, and elementary teachers in particular would preside over a chaotic learning environment.

[S] **Consistency with CCSSM: Materials for science and technical subjects are consistent with CCSSM.** Materials for these subjects in K–8 do not subtract from the focus and coherence of the Standards by outpacing CCSSM math progressions in grades K–8 or misaligning to them. In grades 6–8, materials for these subjects also build coherence across the curriculum and support college and career readiness by integrating key mathematics into the disciplines, particularly simple algebra in the physical sciences and technical subjects, and basic statistics in the life sciences and technical subjects (see Table 3 for a possible picture along these lines).

Table 3

Algebraic competencies integrated into materials for middle school science and technical subjects	Statistical competencies integrated into materials for middle school science and technical subjects
<ul style="list-style-type: none">• Working with positive and negative numbers (including fractions) to solve problems• Using variables and writing and solving equations to solve problems• Recognizing and using proportional relationships to solve problems• Graphing proportional relationships and linear functions to solve problems	<ul style="list-style-type: none">• Working with distributions and measures of center and variability• Working with simple probability and random sampling• Working with bivariate categorical data (e.g., two-way tables)• Working with bivariate measurement data (e.g., scatter plots) and linear models

Indicators of quality in instructional materials and tools for mathematics

The preceding criteria express important dimensions of alignment to the Standards. The following are some additional dimensions of quality that materials and tools should exhibit in order to give teachers and students the tools they need to meet the Standards:

- Problems in the materials are worth doing:
 - The underlying design of the materials distinguishes between *problems* and *exercises*. Whatever specific terms are used for these two types, in essence the difference is that in solving problems, students learn new mathematics, whereas in working exercises, students apply what they have already learned to build mastery. Problems are problems because students haven't yet learned how to solve them; students are learning from solving them. Materials use problems to teach mathematics. Lessons have a few well designed problems that progressively build and extend understanding. Practice exercises that build fluency are easy to recognize for their purpose. Other exercises require longer chains of reasoning.
 - Each problem or exercise has a purpose—whether to teach new knowledge, bring misconceptions to the surface, build skill or fluency, engage the student in one or several mathematical practices, or simply present the student with a fun puzzle.
 - Assignments aren't haphazardly designed. Exercises are given to students in intentional sequences—for example, a sequence leading from prior knowledge to new knowledge, or a sequence leading from concrete to abstract, or a sequence that leads students through a number of important cases, or a sequence that elicits new understanding by inviting students to see regularity in repeated reasoning. Lessons with too many problems make problems a commodity; they forbid concentration, and they make focus and coherence unlikely.
 - The language in which problems are posed is carefully considered. Note that mathematical problems posed using only ordinary language are a special genre of text that has conventions and structures needing to be learned. The language used to pose mathematical problems should evolve with the grade level and across mathematics content.
- There is variety in the pacing and grain size of content coverage.
 - Materials that devote roughly equal time to each content standard do not allow teachers and students to focus where necessary.
 - The Standards are not written at uniform grain size. Sometimes an individual content standard will require days of work, possibly spread over the entire year, while other standards could be sufficiently addressed when grouped with other standards and treated in a shorter time span.
- There is variety in what students produce: Students are asked to produce answers and solutions, but also, in a grade-appropriate way, arguments, explanations, diagrams, mathematical models, etc. In a way appropriate to the grade level, students are asked to answer questions or develop explanations about why a solution makes sense, how quantities are represented in expressions, and how elements of symbolic, diagrammatic, tabular, graphical and/or verbal representations correspond.

- Lessons are thoughtfully structured and support the teacher in leading the class through the learning paths at hand, with active participation by all students in their own learning and in the learning of their classmates. Teachers are supported in extending student explanations and modeling explanations of new methods. Lesson structure frequently calls for students to find solutions, explain their reasoning, and ask and answer questions about their reasoning as it concerns problems, diagrams, mathematical models, etc. Over time there is a rhythm back and forth between making sense of concepts and exercising for proficiency.
- There are separate teacher materials that support and reward teacher study, including:
 - Discussion of the mathematics of the units and the mathematical point of each lesson as it relates to the organizing concepts of the unit.
 - Discussion of student ways of thinking with respect to important mathematical problems and concepts—especially anticipating the variety of student responses.
 - Guidance on interaction with students, mostly questions to prompt ways of thinking.
 - Guidance on lesson flow.
 - Discussion of desired mathematical behaviors being elicited among the students.
- The use of manipulatives follows best practices (see, e.g., *Adding It Up*, 2001):
 - *Manipulatives are faithful representations of the mathematical objects they represent.* For example, colored chips can be helpful in representing some features of rational numbers, but they do not provide particularly direct representations of all of the important mathematics. The opposite of the opposite of red isn't clearly blue, for example, and chips aren't particularly well suited as models for adding rational numbers that are not integers (for this, a number line model may be more appropriate).
 - *Manipulatives are connected to written methods.* “Research indicates that students’ experiences using physical models to represent hundreds, tens, and ones can be effective *if* the materials help them think about how to combine quantities and, eventually, how these processes connect with written procedures.” (*Adding It Up*, p. 198, emphasis in the original). For example, base-ten blocks are a reasonable *model* for adding within 1000, but not a reasonable *method* for doing so; nor are colored chips a reasonable *method* for adding integers. (Cf. standards 1.NBT.4, 1.NBT.6, 2.NBT.7, and 5.NBT.7; these are not the only places in the curriculum where connecting to a written method is important). The word “fluently” in particular as used in the Standards refers to fluency with a written or mental method, not a method using manipulatives or concrete representations.
- Materials are carefully reviewed by qualified individuals, whose names are listed, in an effort to ensure:
 - Freedom from mathematical errors¹⁸
 - Grade-level appropriateness

¹⁸ Sometimes errors in materials are simple falsehoods, e.g., printing an incorrect answer to a problem. Other errors are more subtle, e.g., asking students to explain why something is so when it has been defined to be so.

- Freedom from bias (for example, problem contexts that use culture-specific background knowledge do not assume readers from all cultures have that knowledge; simple explanations or illustrations or hints scaffold comprehension).
- Freedom from unnecessary language complexity.
- The visual design isn't distracting or chaotic, or aimed at adult purchasers, but instead serves only to support young students in engaging thoughtfully with the subject.
- Support for English language learners is thoughtful and helps those learners to meet the same standards as all other students. Allowing English language learners to collaborate as they strive to learn and show understanding in an environment where English is used as the medium of instruction will give them the support they need to meet their academic goals. Materials can structure interactions in pairs, in small groups, and in the large group (or in any other group configuration), as some English language learners might be shy to share orally with the large group, but might not have problem sharing orally with a small group or in pairs. (In addition, when working in pairs, if ELLs are paired up with a student who shares the same language, they might choose to think about and discuss the problems in their first language, and then worry about doing it in English.)

Appendix

The Structure is the Standards

Essay by Phil Daro, William McCallum, and Jason Zimba, February 16, 2012¹⁹

You have just purchased an expensive Grecian urn and asked the dealer to ship it to your house. He picks up a hammer, shatters it into pieces, and explains that he will send one piece a day in an envelope for the next year. You object; he says “don’t worry, I’ll make sure that you get every single piece, and the markings are clear, so you’ll be able to glue them all back together. I’ve got it covered.” Absurd, no? But this is the way many school systems require teachers to deliver mathematics to their students; one piece (i.e. one standard) at a time. They promise their customers (the taxpayers) that by the end of the year they will have “covered” the standards.

In the Common Core State Standards, individual statements of what students are expected to understand and be able to do are embedded within domain headings and cluster headings designed to convey the structure of the subject. “The Standards” refers to all elements of the design—the wording of domain headings, cluster headings, and individual statements; the text of the grade level introductions and high school category descriptions; the placement of the standards for mathematical practice at each grade level.

The pieces are designed to fit together, and the standards document fits them together, presenting a coherent whole where the connections within grades and the flows of ideas across grades are as visible as the story depicted on the urn.

The analogy with the urn only goes so far; the Standards are a policy document, after all, not a work of art. In common with the urn, however, the Standards were crafted to reward study on multiple levels: from close inspection of details, to a coherent grasp of the whole. Specific phrases in specific standards are worth study and can carry important meaning; yet this meaning is also importantly shaped by the cluster heading in which the standard is found. At higher levels, domain headings give structure to the subject matter of the discipline, and the practices’ yearly refrain communicates the varieties of expertise which study of the discipline develops in an educated person.

Fragmenting the Standards into individual standards, or individual bits of standards, erases all these relationships and produces a sum of parts that is decidedly less than the whole. Arranging the Standards into new categories also breaks their structure. It constitutes a remixing of the Standards. There is meaning in the cluster headings and domain names that is not contained in the numbered statements beneath them. Remove or reword those headings and you have changed the meaning of the Standards; you now have different Standards; you have not adopted the Common Core.

Sometimes a remix is as good as or better than the original. Maybe there are 50 remixes, adapted to the preferences of each individual state (although we doubt there are 50 good ones). Be that as it may, a remix of a work is not the same as the original work, and with 50 remixes we would not have common standards; we would have the same situation we had before the Common Core.

Why is paying attention to the structure important? Here is why: The single most important flaw in United States mathematics instruction is that the curriculum is “a mile wide and an inch deep.” This finding comes from research comparing the U.S. curriculum to high performing countries, surveys of

¹⁹ <http://commoncoretools.me/2012/02/16/the-structure-is-the-standards/>.

college faculty and teachers, the National Math Panel, the Early Childhood Learning Report, and all the testimony the CCSS writers heard. The standards are meant to be a blueprint for math instruction that is more focused and coherent. The focus and coherence in this blueprint is largely in the way the standards progress from each other, coordinate with each other and most importantly cluster together into coherent bodies of knowledge. Crosswalks and alignments and pacing plans and such cannot be allowed to throw away the focus and coherence and regress to the mile-wide curriculum.

Another consequence of fragmenting the Standards is that it obscures the progressions in the standards. The standards were not so much assembled out of topics as woven out of progressions. Maintaining these progressions in the implementation of the standards will be important for helping all students learn mathematics at a higher level. Standards are a bit like the growth chart in a doctor's office: they provide a reference point, but no child follows the chart exactly. By the same token, standards provide a chart against which to measure growth in children's knowledge. Just as the growth chart moves ever upward, so standards are written as though students learned 100% of prior standards. In fact, all classrooms exhibit a wide variety of prior learning each day. For example, the properties of operations, learned first for simple whole numbers, then in later grades extended to fractions, play a central role in understanding operations with negative numbers, expressions with letters and later still the study of polynomials. As the application of the properties is extended over the grades, an understanding of how the properties of operations work together should deepen and develop into one of the most fundamental insights into algebra. The natural distribution of prior knowledge in classrooms should not prompt abandoning instruction in grade level content, but should prompt explicit attention to connecting grade level content to content from prior learning. To do this, instruction should reflect the progressions on which the CCSSM are built. For example, the development of fluency with division using the standard algorithm in grade 6 is the occasion to surface and deal with unfinished learning with respect to place value. Much unfinished learning from earlier grades can be managed best inside grade level work when the progressions are used to understand student thinking.

This is a basic condition of teaching and should not be ignored in the name of standards. Nearly every student has more to learn about the mathematics referenced by standards from earlier grades. Indeed, it is the nature of mathematics that much new learning is about extending knowledge from prior learning to new situations. For this reason, teachers need to understand the progressions in the standards so they can see where individual students and groups of students are coming from, and where they are heading. But progressions disappear when standards are torn out of context and taught as isolated events.